Lecture 12

Tuesday, February 23, 2021 3:58 PM

* Prayer

* Spiritual thought

* Answering questions . . .

Directional derivatives:

$$f, \ (x_0, y_0, \ldots), \ \underbrace{\underset{\sim}{u}}_{\text{unit vector}}$$

$$D_u f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

$$= g'(0) \quad \text{where} \quad g(h) = f(x_0 + ha, y_0 + hb).$$

By the chain rule,

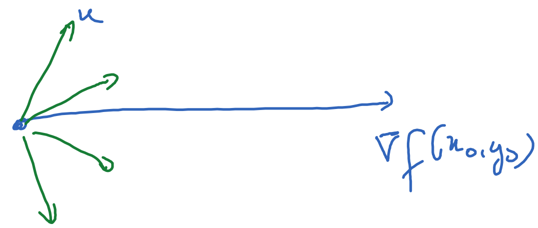
$$g'(h) = a f_x(x_0 + ha, y_0 + hb) + b f_y(x_0 + ha, y_0 + hb)$$

$$g'(0) = a f_x(x_0, y_0) + b f_y(x_0, y_0)$$

$$= \langle a, b \rangle \cdot \underbrace{\langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle}_{\nabla f(x_0, y_0)} \quad \text{(the gradient vector)}$$

$$\boxed{D_u f(x_0, y_0) = u \cdot \nabla f(x_0, y_0)}$$

Largest rate of increase :



$u \cdot \nabla f(x_0,y_0)$ is maximum if u is parallel to $\nabla f(x_0,y_0)$.

$$u = \frac{\nabla f(x_0,y_0)}{|\nabla f(x_0,y_0)|}$$

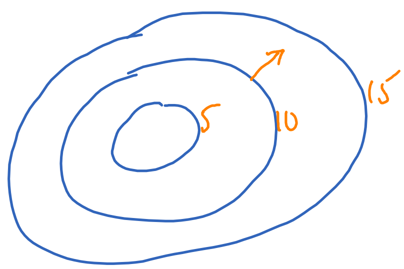
Ex : $f(x,y) = 2xy^2$

$(x_0,y_0) = (2,1)$

what is the direction that has maximum rate of change ?

" " " " least (zero) rate of change ?

Geometrically,



$\boxed{\text{Gradient vector is perpendicular to the level set.}}$

* Application of this principle :

Find tangent vector of a surface : graph of $f(x,y)$

Graph of $f(x,y)$ is the 0-level set of function $g(x,y,z) = z - f(x,y)$.

Normal vector on the graph of f is $\nabla g = \langle -f_x, -f_y, 1 \rangle$.

Ex: Find a normal vector to

① Graph of $f(x,y) = \sqrt{x^2+y^2}$ at point $(4,3,5)$.

② Ellipsoid $x^2+4y^2+5z^2=10$ at $(1,1,1)$.

③ The parabola $y=4x^2+2x+1$ at $(0,1)$.

Extrema of a function

- local extrema
- absolute extrema
- saddle point

If f attains local extrema at a point then $\nabla f = 0$ at that point.

Second derivative test:

$$f(x) = f(x_0) + \underbrace{f'(x_0)}_{0}(x-x_0) + \underbrace{\frac{f''(x_0)}{2}}_{\substack{>0 \\ <0 \\ =0}}(x-x_0)^2 + \cdots$$

$$f(x,y) = f(x_0,y_0) + \underbrace{f_x}_{0}\Delta x + \underbrace{f_y}_{0}\Delta y + \underbrace{\frac{1}{2}f_{xx}(\Delta x)^2 + f_{xy}(\Delta x)(\Delta y) + \frac{1}{2}f_{yy}(\Delta y)^2}_{(\Delta y)^2\left(\frac{1}{2}f_{xx}r^2 + f_{xy}r + \frac{1}{2}f_{yy}\right)} + \cdots$$

$D = f_{xx} f_{yy} - f_{xy}^2$

If $D > 0$, $f_{xx} > 0$: local min

If $D > 0$, $f_{xx} < 0$: local max

If $D < 0$: saddle

If $D = 0$: inconclusive

Ex: $f(x,y) = 3x^2 - x^3 + 2xy + y^2$

Find critical points, local min, local max, abs. min, abs. max, saddle point.